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Soliton Scattering on the External Potential in Weakly Nonlocal Nonlinear Media

Umarov, B.A.*[1] and Busul Aklan, N. A.[2]

[1] *Deparment of Physics, Kulliyyah of Science, International Islamic University Malaysia, 25200 Kuantan, Pahang, Malaysia*
[2] *Department of Computational and Theoretical Sciences, Kulliyyah of Science, International Islamic University Malaysia, 25200, Kuantan, Pahang, Malaysia*

E-mail: bakhram@iium.edu.my

ABSTRACT

The Nonlinear Schrödinger Equation (NLSE) is one of the universal mathematical models and it arises in a such diverse areas as plasma physics, condensed matter physics, Bose - Einstein condensates, nonlinear optics, etc. In this work the scattering of the soliton of the generalized NLSE on the localized external potential has been studied, taking into account the weak nonlocality of the media. We have applied the approximate analytical method, namely the variational method to derive the equations for soliton parameters evolution during the scattering process. The validity of approximations were checked by direct numerical simulations with soliton initially located far from potential. It was shown that depending on initial velocity of the soliton, the soliton may be reflected by potential or transmitted through it. The critical values of the velocity separating these two scenarios have been identified.

Keywords: Soliton, nonlinear equations, scattering, variational methods.

1. Introduction

The investigation of nonlinear wave processes become one of the most interesting problems of the modern physics and mathematics, with numerous applications in different areas of the physics and engineering (Scott, 2005); (Remoissenet, 2003). The localized waves propagating in nonlinear media, or solitons are the objects attracting the attention of the researches, and have been theoretically and experimentally studied first in the context of water waves (Witham, 1974). Then it become clear that soliton is the universal concept and it has been discovered in the solid state physics , nonlinear optics, Bose-Einstein condensates, plasma physics, etc (Bullough and Caudrey, 1980); (Agrawal, 1995); (Pitaevski and Stringari, 2003). The theoretical studies of solitons based on few models, reflecting the basic principles of the physics of nonlinear waves, and also on the generalizations of these models, expending the limits of their applicability. One of this models is the nonlinear Schrödinger equation (NLSE) (Ablowitz et al., 2004). This is the partial differential equation, which is the integrable by Inverse Scattering Transform (IST) method and has multisoliton solutions. Solitons of NLSE preserve their identity while propagating and interacting with each other. The NLSE describes propagation of the wave packets in weakly nonlinear and weakly dispersive medium. For applications it is important to be able to control solitons, which can be achieved by interaction of soliton with another soliton or excitation of the system. Also the soliton can be managed using external perturbations, but in this case it is necessary to modify the model and include additional terms. In this work we study the interaction of the soliton of the generalized NLSE, taking into account a weak nonlocality of nonlinearity, with the localized linear potential.

The paper is organized as follows. In Sec.2 the mathematical model is formulated and basic equations are derived. Sec.3 is devoted to numerical simulations of the variational equations and comparison of the results with direct solution of the original weakly nonlocal NLSE. Finally, in Sec.4 we summarize our findings, and discuss some interesting directions for future studies.

2. The model and main equations

We begin with the discussion of a wave propagation in nonlinear nonlocal weakly dispersive media , and assume that nonlinearity is of the Kerr type. Then the following generalized NLSE equation can be written as a mathematical model of the above mentioned system (Krolikowski and Bang, 2000).

$$i\psi_t + \frac{1}{2}\psi_{xx} + \Delta n(|\psi|^2)\psi + V(x)\psi = 0, \tag{1}$$

where $\psi(x,t)$ is the field function and $V(x)$ is the external potential. In the following we consider the narrow potential which can be modeled by delta function $V(x) = U_0\delta(x)$, and U_0 is the amplitude of the potential. The function $\Delta n = \int_{-\infty}^{\infty} R(x-x')|\psi|^2 dx'$ is the model of nonlocal nonlinearity and $R(x)$ is the response function of nonlocal medium. In case of the $R(x) = \delta(x)$ the response becomes singular, and the Eq.(1) will be reduced to NLSE equation. When the nonlocality is weak, i.e. the response function is narrow in comparison with the localized wave width Eq.(1) can be further reduced to the following generalized NLSE (Krolikowski and Bang, 2000)

$$i\psi_t + \frac{1}{2}\psi_{xx} + (|\psi|^2 + \gamma\partial_x^2(|\psi|^2)\psi + V(x)\psi = 0, \tag{2}$$

where $\gamma = \frac{1}{2}\int_{-\infty}^{\infty} R(x)x^2 dx$ and it is assumed that the response function is normalized $\int_{-\infty}^{\infty} R(x)dx = 1$.

When $V(x) = 0$ Eq.(2) possesses bright stationary soliton solution

$$\psi(x,t) = u(x)\exp(i\Gamma t), \tag{3}$$

and $u(x)$ can be found analytically in implicit form (Krolikowski and Bang, 2000)

$$\pm x = \frac{1}{u_0}\tanh^{-1}\left(\frac{\sigma}{u_0}\right) + \sqrt{4\gamma}\tan^{-1}(\sqrt{4\gamma}\sigma), \tag{4}$$

where u_0 is the maximum of $u(x)$ and $\sigma^2 = (\rho_0 - \rho)/(1 + 4\gamma\rho)$, $\rho = u^2$, $\rho_0 = u_0^2$.

It is easy to verify that the governing Eq. (2) can be obtained from the following Lagrangian density (Bezuhanov et al., 2008)

$$\mathcal{L} = \frac{i}{2}(\psi\psi_t^* - \psi^*\psi_t) + \frac{1}{2}|\psi_x|^2 + V(x)|\psi|^2 - \frac{g}{2}|\psi|^4 + \frac{\gamma}{2}(\partial_x|\psi|^2)^2, \tag{5}$$

by means of the Euler-Lagrange equation.

Now one can apply the variational optimization procedure (Anderson, 1983); (Malomed, 2002) to get the approximate system of ordinary differential equations for soliton parameters. The trial function which approximate the solution of the Eq. (2) we choose from the following considerations. When one neglects the nonlocality and external potential, so that $V(x) = 0$ and $\gamma = 0$, Eq. (2)

will be reduced to the NLSE with well known soliton solution. So, when the external potential and nonlocality are weak, they can be considered as a perturbation, and the shape of solution in this case will remain the same as soliton of NLSE, but the parameters, which are constants in exact solution on NLSE, will be changing according to the variational evolution equations, which will be derived from averaged Lagrangian. Then spatial integration of the Lagrangian density $L = \int_{-\infty}^{\infty} \mathcal{L} dx$ using the trial function

$$\psi(x,t) = A\text{sech}\left(\frac{x-\xi}{a}\right) e^{ib(x-\xi)^2 + iv(x-\xi) + i\varphi}, \tag{6}$$

gives rise to following averaged/effective Lagrangian

$$L = N\left[\frac{\pi^2}{12}a^2 b_t + \frac{\pi^2}{6}a^2 b^2 - \frac{1}{2}\xi_t^2 + \varphi_t + \frac{1}{6a^2} - \frac{N}{6a} - \frac{U_0}{2a}\text{sech}^2\left(\frac{\xi}{a}\right) + \frac{2\gamma N}{a^3}\right]. \tag{7}$$

The norm of the wave function $N = \int_{-\infty}^{\infty} |\psi|^2 dx = 2A^2 a$ is a conserved quantity.

Evolution equations for variational parameters can be derived from the Euler-Lagrange equations $d/dt(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = 0$, where q_i are time dependent collective coordinates a, ξ, b, φ. The equation for the phase φ reduces to $dN/dt = 0$ and illustrates the conservation of the norm of the wave function. It is decoupled from other equations and can be dropped in further analysis. What remains is a set of coupled equations for the width and center-of-mass position of the soliton

$$a_{tt} = \frac{4}{\pi^2 a^3} - \frac{2N}{\pi^2 a^2} + \frac{6U_0}{\pi^2 a^2}\text{sech}^2\left(\frac{\xi}{a}\right)\left[1 - \frac{2\xi}{a}\tanh\left(\frac{\xi}{a}\right)\right] + \frac{24\gamma N}{5\pi^2 a^4}, \tag{8}$$

$$\xi_{tt} = -\frac{U_0}{a^2}\text{sech}^2\left(\frac{\xi}{a}\right)\tanh\left(\frac{\xi}{a}\right). \tag{9}$$

When external potential is absent, i.e. $U_0 = 0$, Eqs.(8) and (9) decouple, and from Eq.(8) one can find the approximate width of the stationary($a_{tt} = 0$) soliton solution of weakly nonlocal NLSE $a_s = \sqrt{12\gamma/5 + 1/A^2}$. Perturbations may generate oscillations of the width around this stationary point. The velocity ξ_t in this case is the constant free parameter. Inclusion of delta potential to the system, couples the time evolution of position of the center of the soliton with the evolution of its width. Obviously when soliton is located far from inhomogeneity, it does not affected by it, and the soliton's parameters are constant. Some qualitative results about solitons evolution we can get, if we neglect the effect of potential to the width of soliton. Then the equation (9) describes the scattering of effective classical particle on the localized barrier.

$$\xi_{tt} = -\frac{U_0}{a_s^2}\text{sech}^2\left(\frac{\xi}{a_s}\right)\tanh\left(\frac{\xi}{a_s}\right) = \frac{dV_P(\xi)}{d\xi}\ . \tag{10}$$

This equation can be integrated once and reduced to the following equation

$$\xi_t = \sqrt{2V_P(\xi)} \ , \qquad (11)$$

where $V_P(\xi) = (U_0/2a_s)\text{sech}^2(\xi/a_s)$ is the effective potential representing the influence of the original localized delta potential to the solitons velocity. From the Eq. (11) it is clear that the effective particle can be transmitted through potential or reflected from it depending on whether the velocity above or below of critical value $v_c = \sqrt{U_0/a_s}$.

3. Numerical simulations

The results obtained above are approximate, and based on some assumptions, so they should be compared with the results of the direct numerical solutions of the governing equations. Numerical solution of the generalized NLSE (2) has been performed by the split-step fast Fourier transform method (Agrawal, 1995) using 2048 Fourier modes within the integration domain of length $L \in [-20 \div 20]$, and the time step was $\delta t = 0.005$. The dynamical equations of the variational approximation (8)-(9) are solved using the Runge-Kutta procedure of 4-th order (Press et al., 1996). Soliton $\psi(x)$ is set in motion with some velocity v towards the potential barrier $V(x)$ initially located at some distance from it. Let us first discuss the results of numerical solutions of the system of ordinary differential equations (8)-(9), describing the approximate time evolution of the width and position of the center of the soliton. Initial width is chosen equal to $a(0) = a_s$ and $A = 1.414241$. As it was discussed before, the soliton behaves like a classical particle with internal degree of freedom associated with the width of soliton. Depending on the amplitude of potential, soliton may be transmitted or reflected from the potential, and when soliton close to the potential, not only its velocity, but also its width will be affected by perturbation. The critical parameters of the system separating transmission from reflection quite well described by the formula, obtained above from simplified 'effective particle' picture of the soliton scattering on the localized weak potential. The examples of the results of numerical solution of the Eqs. (8) & (9) shown in the Figs.(1) & (2)

Also, the examples of results of numerical experiment with the governing generalized NLSE (2), on the same range of parameters and the initial condition chosen in the form of exact soliton from Eq.(3) with $u_0 = A = 1.414241$, and $\Gamma = 1$ are shown on the Fig.(3), which confirm qualitatively that soliton behaves like a particle and can be either transmitted or reflected on the potential wall.

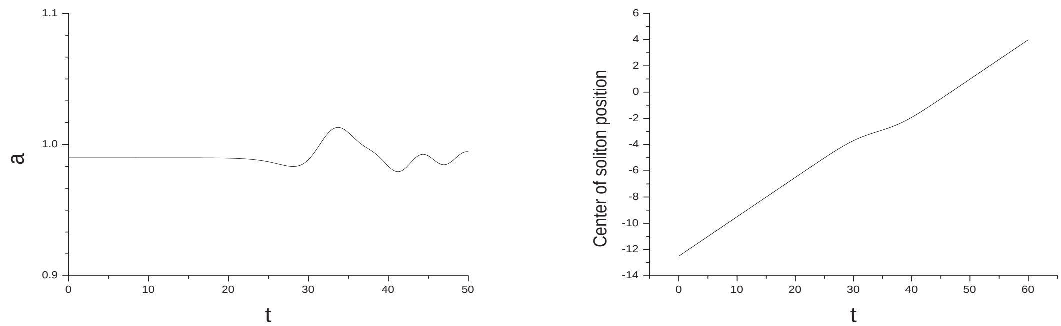

Figure 1: Evolution of the width (left panel) and center of soliton position (right panel) of a soliton in the presence of a delta potential barrier $V(x) = U_0\delta(x+3)$, according to ODE systems: (8)-(9). Parameters: $U_0 = 0.07$, $\gamma = 0.2$, $a = 0.98994$, $v = 0.3$, $\xi_0 = -12.5$.

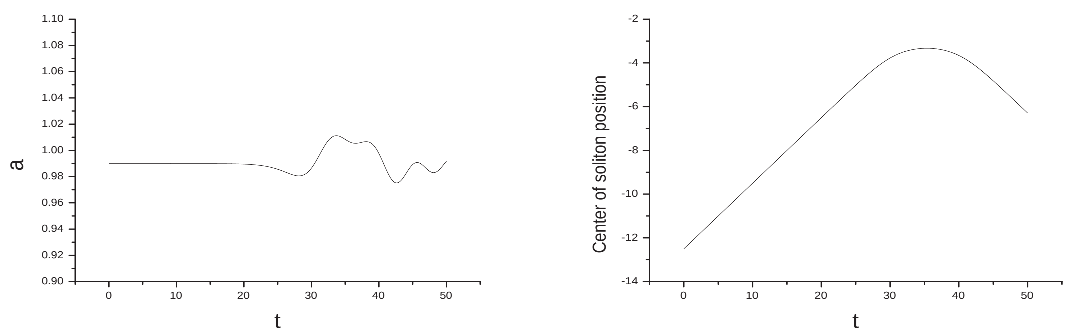

Figure 2: Evolution of the width (left panel) and center of soliton position (right panel) of a soliton in the presence of a delta potential barrier $V(x) = U_0\delta(x+3)$, according to ODE systems: (8)-(9). Parameters: $U_0 = 0.1$, $\gamma = 0.2$, $a = 0.98994$, $v = 0.3$, $\xi_0 = -12.5$.

4. Conclusion

We have developed a variational approximation to describe the scattering of solitons of weekly nonlocal NLSE by external delta potential. Dynamical equations for the parameters of the soliton have the form of ordinary differential equations. Quite good agreement between the results of variational equations and direct numerical solution of the original generalized NLS equation is found for soliton scattering on week potential barrier, when the particle picture can be effectively applied. The future planned research includes the consideration of soliton interaction with localized potential walls as well as potential wells of

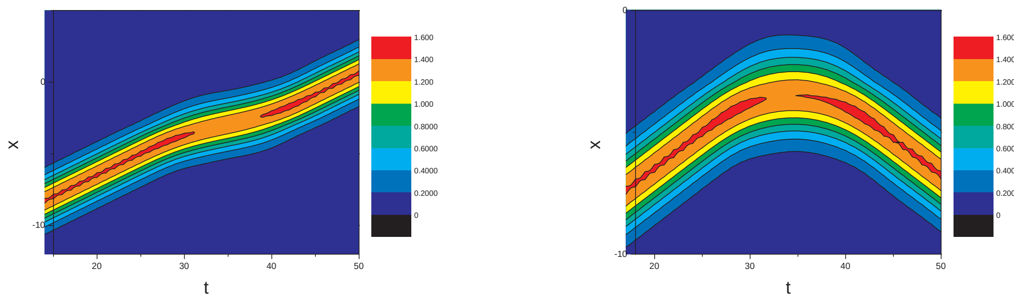

Figure 3: Evolution of the $|\psi(x,t)|$ with soliton initial conditions and $V(x) = U_0\delta(x+3)$, according to the Eq.(2). In the left panel the parameters are the same as in Fig.(1), and in the right panel the parameters are the same as in Fig(2).

the different shapes. The results can be useful in development of new methods aimed at probing the external potentials/defects by scattering solitons on them.

Acknowledgments

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